ON SOME FUNCTIONAL EQUATIONS

(O NEKOTORYKH FUNKTSIONAL'NYKH URAVNENIIAKH)

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1. Many problems of mathematical physics can be reduced to the problem of finding an unknown function X_n (of the integer argument n) which satisfies two functional equations of the form

$$\sum_{n=0}^{\infty} X_n Q_n \varphi_n(x) = f(x) \qquad (0 < x < a)$$
$$\sum_{n=0}^{\infty} X_n R_n \varphi_n(x) = h(x) \qquad (a < x < b) \tag{1}$$

Here, $f(x)$ $(0 \leqslant x \leqslant a)$ and $h(x)$ $(a \leqslant x \leqslant b)$ are given functions of x; Q_n and R_n are known functions of the index n, while $\phi_n(x)$ $(n = 0, 1 \ldots)$ is a system of functions which is complete in $L^2[0, b]$. In a recent work by Cook and Tranter [1] the particular case of Equations (1) is investigated when

$$R_n = 1 \quad (n = 0, 1, \ldots), \qquad Q_n = \alpha_n{}^p \quad (-1 \leqslant p \leqslant 1), \qquad \varphi_n(x) = J_\nu(\alpha_n x)$$

where J_ν is the Bessel function of order ν $(\nu > -1)$, a_n is a positive root of the equation $J_\nu(a_n b) = 0$. In this note we shall consider another special case

$$\sum_{n=0}^{\infty} X_n (1 \pm M_n) P_n(\cos\nu) = f(\nu) \qquad (0 < \nu < \alpha)$$
$$\sum_{n=0}^{\infty} X_n (n + 1/2) P_n(\cos\nu) = h(\nu) \qquad (\alpha < \nu < \pi) \tag{2}$$

where P_n $(\cos \nu)$ are Legendre polynomials. It is assumed that the functions $f(\nu)$ $(0 \leqslant \nu \leqslant a)$ and $h(\nu)$ $(a \leqslant \nu \leqslant \pi)$ have continuous second-order derivatives on the indicated intervals and that the quantity M_n is bounded and decreases (as n approaches infinity) not slower than

$$O(1/n^{2+\varepsilon})\ (\varepsilon > 0)$$

Note that without loss of generality one may assume in (2) that $h(\nu) \equiv 0$ since the case $h(\nu) \not\equiv 0$ can be reduced to the case $h(\nu) \equiv 0$ by a known transformation [2].

In the sequel we give a special method for solving Equation (2) which makes it possible to express X_n in quadratures by means of an auxiliary function which is the solution of a homogeneous Fredholm integral equation with a continuous kernel; hereby we make use of certain ideas presented in [3].

2. We shall try to find a solution of Equations (2) (under the condition that $h(\nu) \equiv 0$) of the form

$$X_n = \int_0^\alpha \psi(\eta) \cos\left(n + \tfrac{1}{2}\right) \eta d\eta \tag{3}$$

where $\psi(\eta)$ is an auxiliary function having a continuous derivative on the interval $[0, \alpha]$. For such a choice of the quantity X_n, the second equation in (2) is satisfied identically. One can easily verify this by integrating by parts the integral in (3) and making use of the next equations [5]:

$$\sum_{n=0}^\infty \sin\left(n + \tfrac{1}{2}\right) \eta P_n(\cos\nu) = \begin{cases} 0 & (0 \leqslant \eta < \nu < \pi) \\ \dfrac{1}{\sqrt{2(\cos\nu - \cos\eta)}} & (0 < \nu < \eta < \pi) \end{cases}$$

In order to find the function $\psi(\eta)$, we substitute Formula (3) into the first equation of (2), and obtain

$$\sum_{n=0}^\infty (1 \pm M_n) P_n(\cos\nu) \int_0^\alpha \psi(\eta) \cos\left(n + \tfrac{1}{2}\right) \eta d\eta = f(\nu) \qquad (0 < \nu < \alpha)$$

But [4, 5]

$$\sum_{n=0}^\infty \cos\left(n + \tfrac{1}{2}\right) \eta P_n(\cos\nu) = \begin{cases} \dfrac{1}{\sqrt{2(\cos\eta - \cos\nu)}} & (0 \leqslant \eta < \nu < \pi) \\ 0 & (0 < \nu < \eta < \pi) \end{cases}$$

Furthermore [4],

$$P_n(\cos\nu) = \frac{2}{\pi} \int_0^\nu \frac{\cos(n + 1/2)\eta d\eta}{\sqrt{2(\cos\eta - \cos\nu)}}$$

Therefore, the first of the equations of (2) will take the form

$$\int_0^\nu \frac{\psi(\eta)\,d\eta}{\sqrt{2(\cos\eta-\cos\nu)}} \pm \frac{2}{\pi}\sum_{n=0}^{\infty} M_n \int_0^\nu \frac{\cos(n+{}^1/_2)\,\eta d\eta}{\sqrt{2(\cos\eta-\cos\nu)}} \int_0^\alpha \psi(t)\cos\left(n+\frac{1}{2}\right) t dt$$

$$= \int_0^\nu \frac{g(\eta)\sec{}^1/_2\,\eta d\eta}{\sqrt{2(\cos\eta-\cos\nu)}} \qquad (0<\nu<\alpha) \tag{4}$$

Here, the function $g(\eta)$ is determined by means of the integral equation

$$\int_0^\nu \frac{g(\eta)\sec{}^1/_2\,\eta d\eta}{\sqrt{2(\cos\eta-\cos\nu)}} = f(\nu) \qquad (0\leqslant\nu\leqslant\alpha) \tag{5}$$

Setting $g(\eta) = G(\tan\eta/2)$, $f(\nu) = F(\tan+\nu/2)$, and making the substitution $\tau = \tan\eta/2$, $s = \tan\nu/2$, we derive the integral equation

$$\int_0^s \frac{G(\tau)\,d\tau}{\sqrt{s^2-\tau^2}} = \frac{F(s)}{\sqrt{1+s^2}} \qquad \left(0\leqslant s\leqslant \tan\frac{1}{2}\alpha\right)$$

from which we obtain $G(\tau)$ by the formula [6]

$$G(\tau) = \frac{2}{\pi}F(0) + \frac{2\tau}{\pi}\int_0^\tau \left(\frac{F'(s)}{\sqrt{1+s^2}} - \frac{sF(s)}{\sqrt{(1+s^2)^3}}\right)\frac{ds}{\sqrt{\tau^2-s^2}} \qquad \left(0\leqslant\tau\leqslant\tan\frac{1}{2}\alpha\right)$$

Let us introduce the notation

$$\sum_{n=0}^{\infty} M_n \cos\left(n+\frac{1}{2}\right) y = K(y)$$

It is obvious that because of the assumption on the nature of the decrease of M_n at infinity, the function $K(y)$ and its derivative will be continuous. With the new notation, Formula (4) can be transformed into

$$\int_0^\nu \frac{d\eta}{\sqrt{2(\cos\eta-\cos\nu)}}\left\{\psi(\eta) \pm \frac{1}{\pi}\int_0^\alpha \psi(t)\,[K(\eta-t)+K(\eta+t)]\,dt - g(\eta)\sec\frac{1}{2}\eta\right\} = 0$$

$$(0<\nu<\alpha)$$

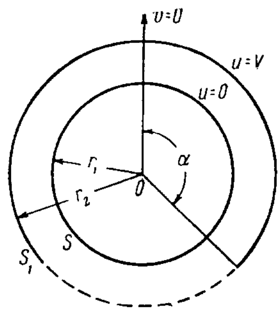
The last equation will be satisfied for all ν if the function $\psi(\eta)$ is a solution of the homogeneous Fredholm integral equation with the continuous kernel $K(\eta - t) + K(\eta + t)$:

$$\psi(\eta) \pm \frac{1}{\pi}\int_0^\alpha \psi(t)\,[K(\eta-t)+K(\eta+t)]\,dt = g(\eta)\sec\frac{1}{2}\eta \qquad (0\leqslant\eta\leqslant\alpha) \tag{6}$$

There exist well-developed methods for solving such equations [7]. If $\psi(n)$ is known, X_n is determined by Formula (3).

We note that the presented formal derivations can all be justified on the basis of the actual properties of ψ.

3. As an example, let us consider a problem in electrostatics. Suppose we are required to find the electric field of a system of conductors which consists of a sphere S and of a non-closed spherical surface S_1; the sphere and surface are assumed to be concentric as shown in the figure. The spherical surface S_1 is charged and has a potential V, the



sphere has zero potential. The determination of the electric field can be reduced, as is well known, to the finding of the potential U satisfying the Laplace equation $\Delta U = 0$ and the boundary conditions

$$U = 0 \quad \text{on } S, \qquad U = V \quad \text{on } S_1, \qquad U = 0 \quad \text{on } \infty$$

We shall look for a solution in spherical coordinates of the form

$$U = \begin{cases} \sum_{n=0}^{\infty} X_n \left(\dfrac{r}{r_2^n} - \beta^{2n+1} \dfrac{r^{-n-1}}{r_2^{-n-1}} \right) P_n(\cos \nu) & (r_1 < r < r_2) \\ \sum_{n=0}^{\infty} X_n (1 - \beta^{2n+1}) \dfrac{r^{-n-1}}{r_2^{-n-1}} P_n(\cos \nu) & (r > r_2) \end{cases} \tag{7}$$

where r and ν are spherical coordinates, r_1 is the radius of the sphere S; r_2 is the radius of the surface S_1, $\beta = r_1/r_2 < 1$ and X_n is the sought function. The function U defined by Equations (7) satisfies formally Laplace's equation and the boundary condition $U = 0$ on S', and is continuous in the entire space including the surface S_1.

From the boundary condition $U = V$ on S_1, and from the condition that the normal derivative of the potential U be continuous on the remaining part of the surface r_2, one can obtain functional equations for the determination of X_n:

$$\sum_{n=0}^{\infty} X_n (1-\beta^{2n+1}) P_n(\cos \nu) = V \qquad (0 < \nu < \alpha)$$
$$\sum_{n=0}^{\infty} X_n \left(n+\frac{1}{2}\right) P_n(\cos \nu) = 0 \qquad (\alpha < \nu < \pi) \tag{8}$$

These equations are particular cases of Equation (2) when

$$M_n = \beta^{2n+1}, \qquad f(\nu) = V, \qquad h(\nu) = 0$$

We have [4,6]

$$K(y) = \beta(1-\beta^2)\frac{\cos {}^1/_2\, y}{1-2\beta^2\cos y+\beta^4}, \qquad g(\eta) = \frac{2V}{\pi}\cos^2\frac{\eta}{2}$$

It is not difficult to show that the function $\psi(n)$ is an even function. The integral equation for its determination, therefore, can be written in the form

$$\psi(\eta) = \frac{\beta(1-\beta^2)}{\pi}\int_{-\alpha}^{\alpha}\frac{\psi(t)\cos {}^1/_2(\eta-t)\,dt}{1-2\beta^2\cos(\eta-t)+\beta^4} + \frac{2V}{\pi}\cos\frac{\eta}{2} \quad (-\alpha \leqslant \eta \leqslant \alpha) \tag{9}$$

Let λ be the first characteristic number of the corresponding homogeneous equation. On the basis of a well-known estimate $|\lambda| \geqslant 1/M$, where

$$M = \max_{-\alpha\leqslant\eta\leqslant\alpha}\int_{-\alpha}^{\alpha}\left|\frac{\cos {}^1/_2(\eta-t)}{1-2\beta^2\cos(\eta-t)+\beta^4}\right| dt = \frac{2}{\beta(1-\beta^2)}\tan^{-1}\left[\frac{2\beta}{1-\beta^2}\sin\frac{1}{2}\alpha\right]$$

we obtain

$$|\lambda| \geqslant \frac{\beta(1-\beta^2)}{2\tan^{-1}\{[2\beta/(1-\beta^2)]\sin {}^1/_2\,\alpha\}}$$

But for all values of β such that $0 < \beta < 1$, we have

$$\frac{\beta(1-\beta^2)}{\pi} < \frac{\beta(1-\beta^2)}{2\tan^{-1}\{[2\beta/(1-\beta^2)]\sin {}^1/_2\,\alpha\}}$$

Hence, Equation (9) is always solvable by the method of successive approximations.

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